International Journal of Theoretical Physics, Vol. 22, No. 1, 1983

Second Quantization, Projective Modules, and Local Gauge Invariance

S. A. Selesnick

Department of Mathematical Sciences, University of Missouri — St. Louis, St. Louis, Missouri 63121

Received May 20, 1981

1. INTRODUCTION

Bundles and bundle structures have gained wide currency in modern approaches to certain topics in quantum physics, significant applications appearing in connection with gauge theories (e.g., Atiyah and Jones, 1978), theories of geometric quantization (e.g., Kostant, 1970; Śniatycki, 1974), and in numerous other contexts. In this paper we argue that such structures can already be discerned in the most elementary notions of second quantization, albeit in disguised form. An examination of the methods traditionally used by physicists in dealing quantum mechanically with systems exhibiting an infinite number of degrees of freedom reveals, almost from the outset, the implicit use of module structures over algebras of functions (Section 2). By making these structures explicit and adapting some results of perturbation theory we arrive at an association between bare particles and finitely generated projective modules (Sections 3 and 4). In particular, rank one modules emerge naturally, for algebraic reasons, as the appropriate descriptors of bosons in this association. (This provides a possible setting for the development of standard geometric quantization theory.) As a first application of the formalism we show the existence of phononlike excitations in general many-fermion systems.

When these ideas are further specialized (local) gauge theoretical notions arise in a natural way out of a consideration of the bundles obtained via Swan's theorem. These theories emerge moreover equipped with an interpretation linked directly to the geometrical entities associated with the underlying bundles. Thus for example in the line bundle (or rank one) case,

0020-7748/83/0100-0029$03.00/0

covariant exterior differentiation defined with respect to a connection on the bundle gives rise to a quantum mechanical version of the Lorentz force law (which now comes with an interpretation in terms of interaction vertices). This allows one to identify the gauge theory arising from a line bundle over an appropriate Lorentz manifold with that of the (homogeneous) Maxwell field (Section 5). These ideas are then extended to the case of the classical Yang–Mills field (Section 6).

Finally (Section 7), we use the formalism to independently derive the usual commutation relations appropriate to the quantum mechanics of a bare boson constrained to one degree of spatial freedom. It is perhaps worth noting that those relations emerge here as a result of relativistic considerations.

2. SECOND QUANTIZATION

The approach to problems involving very large numbers of degrees of freedom known as second quantization has its origin in the interpretation of Schrödinger wave functions as descriptors of a genuine physical entity, some kind of "matter field" for example, rather than merely as vector states in the carrier space of a particular representation of the canonical commutation relations. Thus, under the regime of second quantization, a solution of the ordinary Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t}(\mathbf{x},t)+H\psi(\mathbf{x},t)=0 \tag{1}$$

which can be written

$$\psi(\mathbf{x},t)=\sum_k a_k(t)u_k(\mathbf{x}) \tag{2}$$

in terms of eigenfunctions u_k of H, must itself be "quantized"—that is to say, turned into an operator. The commutation rules for ψ and its canonical conjugate operator (defined by a variational principle, say) must be prescribed on an ad hoc basis. Returning to the expansion (2) one notes that each expansion coefficient $a_k(t)$ can be expressed

$$a_k(t)=\int\psi(\mathbf{x},t)u_k^*(\mathbf{x})d^3x \tag{3}$$

granting that the formal manipulations involved can be perpetrated upon the operator ψ. Thus each $a_k(t)$ must also be considered to be an operator (with commutation relations for different k inherited from those already adopted). Alternatively, one could start with operators $a_k(t)$ and prescribe the canonical variable ψ as an operator through an expansion of the form (2) regardless of its origin. For example (Bjorken and Drell, 1964, p. 26) the free Klein–Gordon equation

$$(\Box + m^2)\phi(x) = 0 \tag{4}$$

has a Fourier integral solution in terms of normalized plane waves f_k of the form

$$\phi(\mathbf{x}, t) = \int \left[a_k f_k(\mathbf{x}, t) + a_k^\dagger f_k^*(\mathbf{x}, t) \right] d^3k \tag{5}$$

and the associated free Klein–Gordon field is quantized when the amplitudes a_k, $a_k^\dagger$ are made into operators (with $a_k^\dagger$ adjoint to a_k) according to the specification

$$a_k = i \int f_k^*(\mathbf{x}, t) \overleftrightarrow{\partial}_0 \phi(\mathbf{x}, t) d^3x \tag{6}$$

where the right-hand side is actually independent of time and the commutation relations for the operator $\phi(\mathbf{x}, t)$ and its canonical conjugate have been assigned.

Another example is afforded by the second quantization of the Dirac field (Bjorken and Drell, 1964, Section 13.2) during the course of which a "field operator" is introduced by

$$\chi(\mathbf{x}, t) = \sum_{\alpha=1}^{N} u_\alpha(\mathbf{x}, t) a_\alpha \tag{7}$$

where the u_α are single-particle wave functions and the a_α are annihilation operators (Bjorken and Drell, 1964, p. 49). The point of this definition is that the single particle wave function $u_{\alpha_i}(\mathbf{x}, t)$ can now be interpreted as the matrix element of the field operator χ between the vacuum state and the single α_i-particle state of the large notional underlying second quantized Hilbert space. Thus a second quantized matrix element is a *function*, not merely a scalar. This interpretation of second quantized *matrix elements* as being identical with first quantized wave *functions* is familiar in field theory and indeed forms the basis for the imposition of LSZ-type asymptotic conditions (Bjorken and Drell, 1964, p. 136).

Now there is a requirement implicit in the types of operator prescriptions exemplified by equation (2) and equation (3): namely, that one should be able to multiply second quantized operators by functions (to obtain new operators) according to the same rules by which one multiplies "first quantized" operators by scalars. That is to say, the second quantized operators are assumed to admit the *linear* (or *module*) *action* of some commutative algebra of functions. Moreover since elements of this algebra can appear as matrix elements of "field" (i.e., second quantized) operators, the implication is that the algebra already acts on a (dense) subspace of the imagined underlying Hilbert space, and it is this action which is inherited by the operators. Now if we think of the transition from first quantization to second quantization as giving rise to the replacement of first quantized matrix elements (constant scalars) by their second quantized analogs [elements in our algebra of functions, according to the discussion following equation (7)] then this transition can be formally effected by assuming that the constant scalars are already contained in the function algebra—or, equivalently, that this algebra contains a unit. Thus we end up with a commutative algebra of functions containing a unit together with a module action on a dense subspace of the Hilbert space of second quantized states. It is the delineation of this apparently neglected structure (which results simply from the extension of the scalars demanded by the most elementary notion of second quantization) and its physical interpretation, which is the subject of the remainder of this paper.

3. PROJECTIVE MODULES

The essential feature of linear spaces when considered as modules over a field is that they are *free*: i.e., they are direct sums of copies of the coefficient field. (This is the module interpretation of the fact that bases exist in a linear space.) Now the module structures implied by the formal algebraic procedures just described in connection with the elementary ideas of second quantization must also be presumed to have this property of being direct sums of copies of the coefficient algebra: for, fixed sets of spanning vectors (e.g., eigenstates of the Hamiltonian, of the number operators, etc.) are retained upon second quantization, the only difference being that one now allows multiplication of members of these sets by elements from a larger class of coefficients. That is to say, tuples of scalar coefficients relative to some basis are simply replaced by tuples of functions (these functions subsequently appearing as matrix elements). This near identity of linear and (implied) module structures allows a direct correspondence between physical conclusions in the two cases to be made unconsciously, since it hides the distinction between the two structures. This is presumably why the module structure has apparently gone unremarked. Now it happens that the class of module structures of physical interest must be enlarged in an apparently slight, but nonetheless crucial, manner. To show this, we consider the case of a free field scattered by some interaction (possibly including self-interactions if asymptotic conditions are assumed) possessing bound states. In terms of perturbation theory one considers the physical incoming field to be a perturbed version of an incoming field which has been "bare" in the remote past and which has achieved its physical or "dressed" status (prior to scattering) as a result of the application of the Møller operator $\Omega^{(+)}$ derived from the interaction dynamical operator $U(t_1, t_2)(t_2 \leqslant t_1)$. Specifically,

$$\Omega^{(+)} \equiv U(0, -\infty) \tag{8}$$

(see Roman, 1965, Chap. 4). Thus, since the unperturbed incoming field is assumed to have no bound states, $\Omega^{(+)}$ maps (a dense subspace of) the Hilbert space $\mathcal{H}_{\text{bare}}$ spanned by the bare incoming states to the Hilbert space $\mathcal{H}_{\text{phys}}$ of true physical incoming states. Again because the unperturbed field has no bound states, this map has a left inverse $\Omega^{(+)*}$ [Roman, 1965, (4-117), (4-130)] and is consequently one-to-one. Thus we may realize an algebraic splitting of vector spaces

$$\mathcal{H}^{\vee}_{\text{phys}} \cong \mathcal{H}^{\vee}_{\text{bare}} \oplus \mathcal{H}^{\vee}_{\text{etc}} \tag{9}$$

where $\mathcal{H}^{\vee}$ denotes a dense subspace of the Hilbert space $\mathcal{H}$, and $\mathcal{H}_{\text{etc}}$ is the space containing the bound states (and possibly others). This splitting is of course completely trivial in the vector space setting, but let us now allow the scalars to be extended to some larger algebra in the manner prescribed by the implicit rules of second quantization. Then, according to these rules, the various linear operators must be presumed to preserve the resulting module structures as well. Consequently (9) holds also for the module structures (MacLane, 1967, Proposition 4-3, p. 16). Now since $\mathcal{H}^{\vee}_{\text{phys}}$ is supposed to describe the actual incoming physical states (which have been prepared from bare states in order to experience scattering at some time $t > 0$) the module structure it carries should be the usual one (this being the one yielding such close agreement with experiment, as in the case of QED for example). That is, $\mathcal{H}^{\vee}_{\text{phys}}$ should be a *free* module over the algebra in question. One can make no such confident claim for the module $\mathcal{H}^{\vee}_{\text{bare}}$ since bare states are of course physically inaccessible; moreover the usual perturbation methods may lead to divergences, owing to the presence of bound states. In terms of (9) these result from the attempt to derive the possibly numerically infinite contributions from $\mathcal{H}_{\text{etc}}$ in the analysis of $\mathcal{H}_{\text{phys}}$, after which such contributions are subtracted (if possible) from the left-hand side to reveal information about $\mathcal{H}_{\text{bare}}$. It is remarkable that as a statement about *modules*, not only does (9) specify $\mathcal{H}^{\vee}_{\text{bare}}$ fairly closely but moreover does so independently of the structure of $\mathcal{H}^{\vee}_{\text{etc}}$. Namely (9) asserts that $\mathcal{H}^{\vee}_{\text{bare}}$, being a direct summand of a free module, is *projective* (see MacLane, 1967, for definitions and elementary properties)—the nature of the other summand ($\mathcal{H}^{\vee}_{\text{etc}}$) is irrelevant to this assertion.

In summary, our algebraic transcription of the idea of renormalization amounts to the replacement of the free module structures already used implicitly to describe dressed states by projective (not necessarily free) module structures to describe bare states. This allows the bare states a considerably richer structure.

4. FINITELY GENERATED PROJECTIVE MODULES

We shall specialize first the module structures themselves, and subsequently the algebra over which they are defined. In elementary quantum mechanics one may associate the notion of a particle with systems exhibiting finite degrees of degeneracy. For instance, particle states may occupy a finite-dimensional subspace of a Hilbert space, coming possibly from an irreducible representation of a compact symmetry group. Following the line of thought put forward above, the analogous property for the module version of this, describing presumably a field of bare particle states, would be that of being *finitely generated* (for definitions and properties of such modules see Bass, 1968, and Swan, 1968). Of course such a module would in general be infinite dimensional as a vector space over the scalar field. Now each finitely generated module M over a commutative algebra A comes equipped with a *rank function*, which is a map

$$rk_M\colon \operatorname{spec} A \to \mathbb{Z}^+,$$

where spec A denotes the set of prime ideals of A and $\mathbb{Z}^+$ the positive integers. (This function is locally constant when spec A is given the hull-kernel topology—cf. Bass, 1968, p. 127.) Now it can be shown that if $\mathfrak{p}_1 \subset \mathfrak{p}_2 (\mathfrak{p}_i \in \operatorname{spec} A, i=1,2)$, then $\operatorname{rk}_M \mathfrak{p}_1 = \operatorname{rk}_M \mathfrak{p}_2$. Consequently the rank function rk_M is determined by its values on maximal ideals. This result shows also that if A is an integral domain rk_M is constant [since $(0) \subset \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{spec} A$]. This is of course trivially the case when A is a field, rk_M in this case coinciding with the dimension of M as a vector space over A. We shall make the simplifying assumption that this constancy of rank extends to the cases we are considering, although without specifying A more closely there seem to be no compelling physical reasons for excluding the more general case, other than the consequences set out below. We shall write rk_M in this case.

Thus, we are reduced to the consideration of finitely generated modules of constant rank (over a commutative unital complex algebra A as yet unspecified) as candidates for the description of bare particle fields. We aim to show first that in this context, rank 1 modules correspond to *bosons*. This follows from the following theorem.

> *Theorem 4.1.* (Swan, 1968, Proposition 8.2, p. 149) Let M be a finitely generated projective module of constant rank n. Then
> (1) $\wedge^i_A M$ is finitely generated and of constant rank for all $i \geqslant 0$,
> (2) $\wedge^n_A M$ has rank 1,
> (3) $\wedge^i_A M = 0$ for $i > n$, where $\wedge_A$ denotes exterior product over A.

In particular if $\mathrm{rk}_M = 1$, (3) gives

$$\wedge^i_A M = 0 \qquad \text{for } i > 1$$

But $\wedge^i_A M$ is the universal object for alternating A-module maps (Lang, 1965, Chap. XVI, Section 6). That is to say, any alternating A-linear map f: $\otimes^i_A M \to P$ factors uniquely through the canonical map $\otimes^i_A M \to \wedge^i_A M$ which is zero in this case. Now any attempt to form an A-module P generated by i-fold antisymmetric tensors of elements from M would give rise to an A-module map sending generators of $\otimes^i_A M$ to generators of P which is alternating (since 2 is not a zero divisor in A) and which necessarily vanishes. Since all such attempts to antisymmetrize generators of $\otimes^i_A M$ fail, these generators must be symmetric. Consequently if they are to represent particle states, the particles must obey Bose–Einstein statistics. It follows that the modules representing (bare) fermion states must have ranks greater than 1.

As our first application we shall consider the case of an infinite system of identical (bare) fermions. From the preceding we are led to consider a finitely generated projective module M of rank >1 to represent the (dense) single bare particle states and the module

$$\begin{aligned}\mathcal{H}^v_{\text{bare}} &= \bigoplus_{i=0}^{\infty} \wedge^i_A M \\ &= A \oplus M \oplus \wedge^2_A M \oplus \cdots \oplus \wedge^{rk_M}_A M \end{aligned} \tag{10}$$

to represent a (second quantized) field of such (noninteracting) bare fermions. This is just an uncompleted version of the usual fermion Fock space construction except that the algebraic operations are conducted over the algebra A. As a vector space over $\mathbb{C}$ ($\subset A$), M is in general infinite dimensional and the direct sum in equation (10) would be nonterminating over $\mathbb{C}$; but as a module over A, it has this novel feature [by (3) of Theorem 4.1]. (One may note that the obvious $\mathbb{C}$-linear map $\wedge_{\mathbb{C}}^{p} M \to \wedge_{A}^{p} M$ is surjective with nontrivial kernel, containing for instance elements of the form $fa \wedge b - a \wedge fb$, $a, b \in M$, $f \in A$. Consequently the above map splits and so, as a $\mathbb{C}$-vector space, $\wedge_{A}^{p} M$ may be realized as a subspace of $\wedge_{\mathbb{C}}^{p} M$, for $p \geq 2$). Moreover the last nonvanishing component in equation (10) has rank one [(2) of Theorem 4.1], signifying the presence in the bare fermion field of excitations having the appearance of free bosons, presumably capable of mediating some kind of "exchange" force between the fermions. If we imagine a fermion–fermion interaction to be switched on, the bare Fock states [equation (10)] will become dressed according to the appropriate analog of (9). But now the bosonlike excitations would themselves partake of the dressing and thereby acquire the appearance of having a mutual interaction. In sum, to the degree of approximation allowed by considering only dense manifolds of states (not to mention the hypotheses of constant rank and finite generation), one would expect an assemblage of physical interacting fermions to be accompanied by a cloud of bosonlike "excitons" which themselves mutually interact. This phenomenon is of course well known in many-body theory, where the excitons are known by various names (phonons, plasmons, etc.) depending on the context. Their bosonlike nature is usually assumed, although certain lengthy approximate calculations (involving assumptions not dissimilar to those above concerning the finiteness of rank) have demonstrated it explicitly in some cases (see Bogoliubov, 1967, p. 221, for the case of metallic phonons).

5. VECTOR BUNDLES AND GAUGE INVARIANCE

We now turn to the choice of algebra A for a free relativistic field. To help motivate such a choice, let us consider the situation analogous to the one described by equation (3) in traditional quantum field theory. Here it has long been realized that physical meaning cannot be ascribed to individual field operators defined at points of space-time, but only to suitably smeared versions of these notional pointwise field operators. This can be expressed figuratively in the form

$$\phi(f) = \int \phi(x) f(x) d^4x \tag{11}$$

where $\phi(x)$ denotes the pointwise field operator, where the underlying Hilbert space may be decomposed as a direct integral over energy-momentum, and where f is chosen to be an infinitely differentiable function of the space-time point x, of compact support (see for example Segal, 1963, p. 37) whose Fourier transform is of sufficiently rapid decrease to ensure that $\phi(f)$ is well defined as an operator on some dense domain in the Hilbert space. Here as in Section 2 there is an implied module action of the algebra of smooth functions on space-time (with pointwise operations) upon at least the dense domain specified. As well as the test functions f, our candidate for the algebra A must include also the constant functions on the presumed space-time manifold. Consequently we are led to consider as a simple possible candidate for A, the algebra of infinitely differentiable (complex) functions on a compact Minkowskian space-time (i.e., a compact simply connected, complete, proper Lorentzian 4-manifold with zero curvature), which corresponds physically to an empty finite universe. This choice of algebra would incorporate the test functions as well as the constants. We shall pursue some local consequences of the preceding formal discussion in this case while noting that compact space-times contain timelike loops which violate global causality: this is sometimes considered objectionable in general relativity. [Other choices of underlying manifold, possibly more reasonable physically, include S^4 considered as a conformal compactification of Minkowski space-time (cf. Atiyah and Jones, 1978), which will be taken up elsewhere. We note that if compactness is relinquished in the subsequent argument the manifold in question could be taken to be Minkowski space itself, with world lines of magnetic monopoles removed. The local assertions in the remainder of the paper would not then be affected.]

Thus, according to the preceding argument, to describe bare particle states operated upon field observables smeared by infinitely differentiable complex test functions in an empty finite universe, we need to consider finitely generated projective modules over the algebra $C^\infty(S)$ of infinitely differentiable complex functions on a manifold S of the type described above. Now, a theorem due primarily to Swan (see Bass, 1968; Swan, 1968) asserts that each such module can be realized as the module of global C^∞ sections of a uniquely determined complex differentiable vector bundle over S with fiberwise dimension (over a point in S) corresponding to the rank function evaluated at the corresponding maximal ideal in $C^\infty(S)$. (For proof of a more general statement see Mulvey, 1976.) Thus we arrive at the study of complex vector bundles over a differentiable manifold, whose sections are to be interpreted as possible bare particle states. We note that in this association, line bundles (i.e., bundles with one-dimensional fiber) correspond to boson fields.

We require now to identify the bundle(s) associated for example with bare fermions. To this end we note that since our manifold admits a proper

Lorentzian metric, the tangent bundle has transition matrices (in its local coordinate description) belonging to the proper Lorentz group. Consequently transition matrices for the cotangent bundle (i.e., the dual of the tangent bundle, identifiable with the bundle of differentiable 1-forms), being representable as the inverse transposed matrices of those used for the tangent bundle, also belong to this group. This is usually expressed in the physical literature by saying that the four-vector dx^{μ} transforms like the coordinate vector x^{μ}, as we transform from one Lorentz frame—i.e., local coordinate patch of our manifold—to another. Thus the global sections of the complexified cotangent bundle (i.e., the bundle of complex valued 1-forms, which we shall denote by Ω^1) can be described locally as sets of four-vectors of complex differentiable functions defined in each frame, which transform according to the transition matrices of Ω^1: that is via proper Lorentz transformations. Specifically, we may write

$$\Lambda_{(\alpha\beta)}\psi_{(\alpha)} = \psi_{(\beta)} \tag{12}$$

where $\psi_{(\alpha)}$ (respectively, $\psi_{(\beta)}$) represents a four-vector of complex differentiable functions defined relative to the Lorentz frame $O_{(\alpha)}$ (respectively $O_{(\beta)}$) and $\Lambda_{(\alpha\beta)}$ represents the appropriate Lorentz transformation relating these frames via

$$\Lambda_{(\alpha\beta)}\mathbf{x}_{(\alpha)} = \mathbf{x}_{(\beta)} \tag{13}$$

where $\mathbf{x}_{(\alpha)} = (x^{\mu})_{(\alpha)}$ etc., denote four-vectors relative to the frame $O_{(\alpha)}$. A family of four-vector "wave" functions $\{\psi_{(\alpha)}\}$ satisfying equation (12) now represents a global section of Ω^1. If this bundle (or strictly its module of sections) is to represent a bare particle it should be possible to perform a dressing perturbation on the four-component wave functions satisfying equation (12) to arrive at a description of the corresponding physical particle. Accordingly, let us suppose observer $O_{(\alpha)}$ applies some dressing perturbation $\Omega^{(+)}_{(\alpha)}$ to a four-component wave function $\psi_{(\alpha)}$ to obtain $\Omega^{(+)}_{(\alpha)}\psi_{(\alpha)}$. If this dressing operation is to be Lorentz covariant we must have

$$\Lambda_{(\alpha\beta)}\Omega^{(+)}_{(\alpha)}\psi_{(\alpha)} = \Omega^{(+)}_{(\beta)}\psi_{(\beta)} \tag{14}$$

or

$$\Omega^{(+)*}_{(\beta)}\Lambda_{(\alpha\beta)}\Omega^{(+)}_{(\alpha)}\psi_{(\alpha)} = \psi_{(\beta)} \tag{15}$$

in the absence of bound states for the unperturbed system. Thus, writing equation (15) as

$$V(\Lambda_{(\alpha\beta)})\psi_{(\alpha)} = \psi_{(\beta)} \tag{16}$$

the effect, as far as the bare states are concerned when such a perturbation is switched on, is equivalent to the replacement of the transformation law equation (12) by one in which the Lorentz transformation (the Λ's) acts according to some *representation* $V(\Lambda)$ of the proper Lorentz group. Now, the condition for the family of locally defined Dirac operators

$$\not\nabla_{(\alpha)} = \gamma^{\mu}\frac{\partial}{\partial x^{\mu}_{(\alpha)}} \qquad \text{(Einstein summation convention)}$$

to be covariant is precisely that wave functions satisfying equation (16), that is the wave functions representing the unknown physical particles we are seeking, should be carried by these operators to wave functions which also represent physical particles—i.e., also satisfying equation (16). Thus for ψ's satisfying equation (16) we should have

$$V(\Lambda_{(\alpha\beta)})(\not\nabla_{(\alpha)}\psi_{(\alpha)}) = \not\nabla_{(\beta)}\psi_{(\beta)}$$

In view of equation (16) this becomes

$$V(\Lambda_{(\alpha\beta)})\not\nabla_{(\alpha)} = \not\nabla_{(\beta)}V(\Lambda_{(\alpha\beta)})$$

or

$$V(\Lambda_{(\alpha\beta)})\Lambda_{(\alpha\beta)}\not\nabla_{(\beta)} = \not\nabla_{(\beta)}V(\Lambda_{(\alpha\beta)})$$

But the equation

$$V(\Lambda)\Lambda\not\nabla = \not\nabla\, V(\Lambda)$$

specifies $V(\Lambda)$ precisely as the spinor representation (see Bjorken and Drell, 1965, Section 2.2). Consequently a Lorentz covariant dressing of the states represented by the sections of Ω^1 gives rise to wave functions which transform as Lorentz spinors and hence are appropriate to the description of physical spin-1/2 fermions. Switching off the perturbation then reveals the sections of Ω^1 as having been appropriate to the description of single bare (incoming) spin-1/2 fermion states.

Let us suppose now that single bare bosons are present. In our picture such states are represented by a complex (differentiable) line bundle, L say, over the manifold S (see section 4 above). Along with such a bundle comes certain standard geometrical items to which, in view of the preceding, we can now give physical interpretations. Thus, for such a bundle there always exist *differentiable connections* (or *covariant derivatives*). These can be viewed as additive bundle maps

$$D\colon L \to L\otimes\Omega^1$$

satisfying

$$D(sf) = sDf + f \otimes ds$$

where s is a complex differentiable function on S, f is a section of L, and d denotes exterior differentiation (the latter producing sections of Ω^1). The D's are of course not unique but can be expressed locally, relative to some trivialization $\{U_{(\alpha)}\}_\alpha$, in the form

$$D_{(\alpha)} = d_{(\alpha)} + \psi_{(\alpha)} \tag{17}$$

for $\psi_{(\alpha)} \in \Gamma(U_{(\alpha)}, \mathrm{Hom}(L, L \otimes \Omega^1)) \cong \Gamma(U_{(\alpha)}, \Omega^1)$, and where $d_{(\alpha)}$ represents exterior differentiation applied to local differentiable sections of the now trivial bundle $L | U_{(\alpha)}$ when these are realized as ordinary differentiable functions on $U_{(\alpha)}$. (Here Γ denotes the section functor and $|$ denotes restriction.) Having chosen such a D there is defined a bundle map

$$D^1 \colon L \otimes \Omega^1 \to L \otimes \Omega^2$$

(where $\Omega^p = \Lambda^p \Omega^1$) satisfying

$$D^1(f \otimes \omega) = f \otimes d^1\omega + Df \wedge \omega$$

where d^1 is exterior differentiation $d^1 \colon \Omega^1 \to \Omega^2$ (a bundle map). Now it is easily seen that

$$D^1 D \colon L \to L \otimes \Omega^2 \tag{18}$$

is in fact a *linear* map of bundles and since

$$\mathrm{Hom}(L, L \otimes \Omega^2) \cong \mathrm{Hom}(L, L) \otimes \Omega^2 \cong \Omega^2$$

$D^1 D$ corresponds to a global 2-form, Φ_L say. Thus, we may write

$$D^1 D(f) = f \otimes \Phi_L \tag{19}$$

for a global section f of L. This 2-form has further the property of being closed and consequently it represents a certain cohomology class $[\Phi_L]$ in $\check{H}^2(S, \mathbb{C})$, the second Čech (say) cohomology group of S, by de Rham's theorem. This class is independent of the choice of D and $[(1/2\pi i)\check{\Phi}_L]$ is in fact an *integral* class [i.e., it is in the image of the map $\check{H}^2(S, \mathbb{Z}) \to \check{H}^2(S, \mathbb{C})$ induced by inclusion $\mathbb{Z} \to \mathbb{C}$]. One may choose D so that the ψ's (and consequently the corresponding Φ_L) are Hermitian: in this context this means that $\psi+\bar{\psi}=0$. (For these and other facts see, for example, Vaisman, 1973.) Moreover, Φ_L being closed, we have

$$d^2\Phi_L=0 \tag{20}$$

Choosing an Hermitian connection, and writing Φ_L in the form

$$\Phi_L=ig\big(-E_1dx^0\wedge dx^1-E_2dx^0\wedge dx^2-E_3dx^0\wedge dx^3$$

$$+B_1dx^2\wedge dx^3-B_2dx^1\wedge dx^3+B_3dx^1\wedge dx^2\big) \tag{21}$$

where g is some real constant, equation (20) is recognized as being equivalent to one half of Maxwell's homogeneous equations, namely,

$$\nabla\wedge\mathbf{E}=-\dot{\mathbf{B}}$$

and

$$\nabla\cdot\mathbf{B}=0$$

Thus our considerations have led to the main ingredients of the prototype Abelian gauge theory, the local gauge group in this case being the structure group of the bundle L, which may always be reduced to $U(1)$. Moreover the bundle formalism now has a direct physical interpretation. Thus consider (18) and equation (19). Since D^1D is a bundle map, boson states represented by L are mapped to states represented by $L\otimes\Omega^2$. But this bundle represents boson states combined with states representing pairs of (bare spin-1/2) fermions. Now clearly equation (19) expresses a version of the classical Lorentz law of force in coordinate free, exterior differential form for an appropriate choice of g. To see this connection explicitly we shall consider, rather heuristically, the case of a "wave packet" (a section of L) compactly supported in some neighborhood of the origin of some local frame and defined there by the function

$$f(x^\mu)=\exp(-s)$$

where

$$s^2=x_\mu x^\mu$$

(Such a locally defined function can always be extended to a C^∞ section of L.)

Then locally (dropping the parenthesized subscripts),

$$D(f)=\frac{\partial f}{\partial x^{\mu}}dx^{\mu}+\psi(f)$$

$$=-f(x_{\mu}/s)dx^{\mu}+\psi(f)$$

where ψ is the map defining the connection form [equation (17)]. Now for s sufficiently small, $x_{\mu}/s \approx p_{\mu}$, the associated energy-momentum four-vector, and $f \approx 1$. That is to say, in a small enough neighborhood of the origin of this frame we may write

$$D(1)=-p_{\mu}dx^{\mu}+\psi(1)$$

Then

$$D^{1}D(1)=-Dp_{\mu}\wedge dx^{\mu}+\Phi_{L}$$

Consequently

$$Dp_{\mu}\wedge dx^{\mu}=0$$

or

$$dp_{\mu}\wedge dx^{\mu}=-p_{\mu}\psi_{\nu}dx^{\nu}\wedge dx^{\mu} \tag{22}$$

from equation (17), where $\psi_{\nu}dx^{\nu}$ is the 1-form representing ψ. Quantizing the right-hand side of this equation according to the usual rule in which p_{μ} is replaced by $i\partial/\partial x^{\mu}$ we obtain

$$dp_{\mu}\wedge dx^{\mu}=-i\frac{\partial\psi_{\nu}}{\partial x^{\mu}}dx^{\nu}\wedge dx^{\mu}$$

But the Lorentz law of force may be written [Misner et al., 1973, p. 73, equation (3.4)]

$$dp_{\mu}=eF_{\mu\nu}dx^{\nu}$$

where $F_{\mu\nu}$ is the usual skew-symmetric electro-magnetic tensor. The previous equation then reads

$$eF_{\mu\nu}dx^{\nu}\wedge dx^{\mu}=-i\frac{\partial\psi_{\nu}}{\partial x^{\mu}}dx^{\nu}\wedge dx^{\mu}$$

Equating coefficients of $dx^\nu \wedge dx^\mu$ we obtain

$$
\begin{aligned}
2eF_{\mu\nu} &= -i\left(\frac{\partial\psi_\nu}{\partial x^\mu} - \frac{\partial\psi_\mu}{\partial x^\nu}\right)\\
&= -i(igF_{\mu\nu}) \qquad \text{from equation (21)}\\
&= gF_{\mu\nu}
\end{aligned}
$$

Thus $g=2e$ and we may write

$$
\begin{aligned}
\Phi_L &= ig\tfrac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu\\
&\equiv ieF
\end{aligned}
$$

Conversely, with this choice of Φ_L and upon quantization, we recover from equation (22) [and as a consequence of equation (19)] the Lorentz law of force in the form

$$dp_\mu \wedge dx^\mu = eF_{\mu\nu}dx^\nu \wedge dx^\mu$$

Recalling that Φ_L also represents the bundle map D^1D: $L \to L\otimes\Omega^2$ we have the following interpretation of the above relations: Each boson state (i.e., section of L) gives rise by the "Lorentz force law" prescription (i.e., the map D^1D) to a state representing the interaction of two fermions with a boson (i.e., a section of $L\otimes\Omega^2$)—or, in the language of Feynman graphs, a two-fermion, one-boson interaction vertex. This rather precise interpretation should be compared with the physical notion of a force being mediated between fermions by the exchange of bosons.

The fact that $[eF/2\pi]$ is an intergral class implies that charge is quantized. This has been noted (Śniatycki, 1974) in the context of geometric quantization theory to which our considerations are related in the following way. The real contangent bundle $\Omega^1_{\mathbf{R}}$ over S has itself the structure of a symplectic manifold (with an appropriately chosen symplectic form). Starting with our complex line bundle L on S we may pull L along the projection ρ: $\Omega^1_{\mathbf{R}} \to S$ to obtain a line bundle ρ^*L over the manifold $\Omega^1_{\mathbf{R}}$. The closed 2-form associated with this bundle may be taken to be the pullback $\rho^*\Phi_L$ of Φ_L and consequently $e\rho^*F/2\pi$ represents an integral class in $\check{H}^2(\Omega^1_{\mathbf{R}},\mathbf{C})$.

The fact that $[eF/2\pi]$ is an intergral class implies that charge is quantized. This has been noted (Śniatycki, 1974) in the context of geometric quantization theory to which our considerations are related in the following

Now we note that if A_μ denotes a locally defined electromagnetic four-vector potential, the local $\psi_{(\alpha)}$ in equation (17) may be taken to be

$$\psi_{(\alpha)} = -2ieA_\mu dx^\mu_{(\alpha)}$$

[since then $d^1\psi_{(\alpha)} = D^1D(1)|_{(\alpha)} = \Phi_L|_{(\alpha)}$] and consequently

$$\begin{aligned} D_{(\alpha)} &= d_{(\alpha)} - 2ieA_\mu dx^\mu_{(\alpha)} \\ &= dx^\mu \frac{\partial}{\partial x^\mu} - 2ieA_\mu dx^\mu|_{(\alpha)} \\ &= (\partial_\mu - 2ieA_\mu)dx^\mu|_{(\alpha)} \end{aligned}$$

Thus the connection D is seen to be precisely the contracted differential version of the "minimal coupling" (of electrons to photons) of electrodynamics. This is entirely consistent with our preceding interpretation of this interaction in terms of the underlying bundles themselves. The usual forms of local gauge covariance for the minimal coupling now appear as a consequence of the presence of an underlying bundle (or projective module). Similarly, the gauge covariance of F itself follows from the fact that it is a global section of the bundle $\Omega^2[\cong \mathrm{Hom}(L, L\otimes\Omega^2)]$. See Vaisman (1973), Section 5.3. Since it is gauge covariant, the usual Lagrangian density, which may be written

$$\mathfrak{L} = -\tfrac{1}{4}F \wedge {}^*F$$

$^*F$ denoting the Hodge dual of F with respect to the Lorentz metric, is gauge invariant. Variation of $\mathfrak{L}$ leads to the other pair of Maxwell's homogeneous equations in the form

$$d^{2*}F = 0$$

Now the field, in so far as it is determined by Maxwell's equations, involves only states associated with a two-fermion vertex, according to our interpretation of the 2-form representing D^1D. Moreover, the only higher-order fermion vertices which can arise are those of even order built up from antisymmetric combinations of two-fermion vertex states. In our context, the only possible such vertex is the four-fermion vertex represented by the 4-form

$$(ie)^2 F \wedge F = -(ie)^2 2\mathbf{E}\cdot\mathbf{B}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

which vanishes for free solutions of the Maxwell equations. Consequently, the description of this minimal (bare) electron–electron interaction must be contained entirely in the first two orders of perturbation via Feynman graphs, since no higher-order vertices contribute. This accords well with experiment even for the electromagnetic interaction of dressed particles for which the low-order graphs already give close agreement with experiment (Scadron, 1979, Chap. 11). In terms of a perturbation expansion via Feynman graphs this would seem to indicate that contributions from the second and higher powers of the expansion parameter $e^2/4\pi$ (corresponding to contributions from fourth- and higher-order graphs) can be neglected, whereas the term involving this parameter to first power must be retained. One notes that this property, which characterizes the "weakness" of the interaction (i.e., the "smallness" of e) has been obtained without reference to the actual numerical value of the coupling constant e. Indeed it is independent of this value, originating rather in the geometry of the underlying bundle.

If one were now to proceed with the traditional canonical quantization procedure, the Lagrangian density $\mathfrak{L}$ would be varied with respect to the independent dynamical variables A_μ, after which the A_μ and their resulting canonical conjugates would be treated as operators obeying externally imposed commutation relations. These commutation relations are traditionally copied from those already assumed for the case of a finite number of degrees of freedom. We shall show in Section 6 that our formalism yields an irreducible representation of these commutation relations for a boson constrained to one degree of spatial freedom thereby showing that this essential feature of elementary quantum mechanics may actually be derived ab initio from our algebraic interpretation of second quantization (which did not itself involve any assumptions concerning commutation relations) together with some relativistic considerations. Before doing so, however, we digress briefly to extend some of the preceding ideas to the case of the prototype non-Abelian gauge theory, namely, that of Yang and Mills (1954).

6. THE YANG–MILLS FIELD

Recall that the physical basis for the adoption of the isotopic spin formalism was the observed similarities between the neutron and proton in the absence of electromagnetic effects. This led to the idea that these entities represented two-particle states of a single "nucleon" field, requiring a pair of spinors for its description and obeying some higher symmetry – the breaking of which (for instance by the switching on of the electromagnetic field) was held responsible for the appearance of two-particle states with slightly different masses. In our picture this would amount locally to the replacement of Ω^1 by $\Omega^1 \oplus \Omega^1$, the two components representing the two (bare) nucleon components. Globally, this entails the replacement of the line bundle L in the "interaction" bundle $L \otimes \Omega^1$ (locally isomorphic to Ω^1) by a two-dimensional bundle B($B \otimes \Omega^1$ being locally isomorphic to $\Omega^1 \oplus \Omega^1$). By analogy with the case considered in the last section, we are led to consider connections

$$D: B \to B \otimes \Omega^1$$

which are to be interpreted as contracted versions of a minimal coupling of the B field to the bare (complex) nucleon field. The associated bundle map

$$D^1 D: B \to B \otimes \Omega^2$$

now appears as a 2×2 matrix of 2-forms, by virtue of the bundle isomorphism $\mathrm{Hom}(B, B \otimes \Omega^2) \cong \mathrm{Hom}(B, B) \otimes \Omega^2$. The local gauge group in this case should appear as a subgroup of the structure group of B which can always be reduced to the unitary group—$U(2)$ in this case. As is well known the choice of the isotopic gauge group $SU(2)$ is consistent with the physical facts. Our bundle B should then represent states of the pion field mediating an interaction between the nucleons, in precise analogy with discussion in the preceding section. In this case the connection D is given locally by an equation similar to that of equation (7) of Yang and Mills (1954). That is, locally (dropping the parenthesized subscripts used earlier)

$$\begin{aligned} D &= d_\mu \pm ig(\mathbf{b}_\mu \cdot \boldsymbol{\tau})\,dx^\mu \\ &= (\partial_\mu \pm ig(\mathbf{b}_\mu \cdot \boldsymbol{\tau}))\,dx^\mu \end{aligned}$$

where each $\mathbf{b}_\mu$ is an (isotopic) 3-vector field, $\boldsymbol{\tau}$ is the triple of Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and g is a coupling constant.

Now in general for a connection expressed locally in the form

$$D = d + \omega$$

where locally

$$\omega: B \to B \otimes \Omega^1$$

is a matrix of 1-forms, it is easy and standard that the matrix of 2-forms representing $D^1 D$ may be expressed locally in the form

$$D^1 D = d^1 \omega - \omega \wedge \omega$$

where the exterior operations are performed matrixwise. For a matrix of 1-forms expressed in the form

$$\eta = (\phi_\mu \cdot \boldsymbol{\tau})\,dx^\mu$$

a tedious calculation shows that

$$d^1\eta - \eta \wedge \eta = -\left\{\left[\left(\frac{\partial \phi_\mu}{\partial x^\nu} - \frac{\partial \phi_\nu}{\partial x^\mu}\right) + 2i\phi_\mu \wedge \phi_\nu\right] \cdot \boldsymbol{\tau}\right\} dx^\mu \wedge dx^\nu$$

So with

$$\omega = -ig(\mathbf{b}_\mu \cdot \boldsymbol{\tau})\,dx^\mu$$

we obtain as a local representation of the matrix of 2-forms $D^1D = \Phi_B$,

$$d^1\omega - \omega \wedge \omega = ig\left\{\left[\left(\frac{\partial \mathbf{b}_\mu}{\partial x^\nu} - \frac{\partial \mathbf{b}_\nu}{\partial x^\mu}\right) + 2g\mathbf{b}_\mu \wedge \mathbf{b}_\nu\right] \cdot \boldsymbol{\tau}\right\} dx^\mu \wedge dx^\nu$$

Thus locally

$$\Phi_B = ig(\mathbf{f}_{\mu\nu} \cdot \boldsymbol{\tau})\,dx^\mu \wedge dx^\nu \tag{23}$$

where the $\mathbf{f}_{\mu\nu}$ are precisely the Yang–Mills field strengths if $g = -\varepsilon$ [Yang and Mills, 1954, equation (9)]. Since Φ_B is in fact globally defined as a bundle map

$$\Phi_B \colon B \to B \otimes \Omega^2$$

it commutes with the action of the structure group of B: that is to say, is covariant under local gauge transformations. Consequently the Lagrangian density (for the B field)

$$\mathfrak{L} = -\tfrac{1}{4}\mathbf{f}_{\mu\nu} \cdot \mathbf{f}^{\mu\nu}$$

is gauge invariant. However, in this case since gauge invariance of the Lagrangian prevents the appearance of mass terms, the field equations arising from this or any other gauge invariant Lagrangian cannot specify the (massive) π-meson field completely (cf. Taylor, 1979). Thus we cannot make assertions concerning the strength (or weakness) of the π-mesic interaction analogous to those made earlier for the electromagnetic interaction. The problem of the missing mass terms must therefore be addressed by invoking some symmetry-breaking mechanism such as that of Higgs (Taylor, 1979). This and other more exotic forms of symmetry breaking (which involve the characteristic classes of the bundles whose connections are specified by solutions of the Yang–Mills field equations) may be conducted at the bundle level and will be considered elsewhere.

7. COMMUTATION RELATIONS

In this concluding section we show that the formalism developed above leads naturally to irreducible representations of the usual commutation relations for a boson restricted to one degree of spatial freedom. Covariant suppression of two degrees of spatial freedom in the considerations of Section 5 leads to the choice of a compact two-dimensional Minkowskian space-time (i.e., a compact, simply connected, proper, flat Lorentz manifold of dimension 2) to support the line bundles whose sections are to represent boson states. Now a theorem of Kuiper (Kuiper, 1953, p. 79–87) asserts that the 2-torus T^2 is the only such manifold, so we are reduced to a consideration of line bundles on this space. These have been studied and the results may be described as follows (cf. Selesnick, 1979). If T^2 is realized as $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ is embedded for simplicity as the standard lattice in $\mathbb{R}^2$, then there is an isomorphism between the group of equivalence classes of line bundles on T^2 (with group operation induced by tensor product of fibers) and the group ($\cong\wedge^2\mathbb{Z}^2\cong\mathbb{Z}$) of alternating real-valued bilinear forms on $\mathbb{R}^2$ which are integer valued on $\mathbb{Z}^2$. [For a line bundle L the corresponding σ can be realized as the element in $\check{H}^2(T^2,\mathbb{C})$ determined by D^1D for some connection D on L. As such σ can also be expressed in differential form via DeRham's theorem as the class represented by a constant multiple of $dx^0\wedge dx^1$ (cf. Selesnick, 1979, Proposition 3.1).] The $C^\infty(T^2)$-module of sections corresponding to some σ may be described explicitly as follows. Choose any function

$$F\colon \mathbb{Z}^2\to\mathbb{R}$$

satisfying the congruence

$$F(\mathbf{m}+\mathbf{n})=F(\mathbf{m})+F(\mathbf{n})+\sigma(\mathbf{m},\mathbf{n}) \bmod 2$$

for $\mathbf{m},\mathbf{n}\in\mathbb{Z}^2$. Then the $C^\infty(T^2)$-module of sections of the bundle corresponding to σ is isomorphic with the module of C^∞ functions $f\colon \mathbb{R}^2\to\mathbb{C}$ satisfying the relation

$$f(\mathbf{x}+\mathbf{n})=f(\mathbf{x})\exp \pi i\left[F(\mathbf{n})+\sigma(\mathbf{x},\mathbf{n})\right]$$

for any $\mathbf{x}\in\mathbb{R}^2,\mathbf{n}\in\mathbb{Z}^2$.

Now the C^∞ sections of any bundle over a compact manifold equipped with an appropriate measure may be densely embedded (as a $\mathbb{C}$-vector space) into an essentially unique separable Hilbert space: one chooses a unitary structure in the bundle, forms the usual L^2 inner product on the space of sections and then completes the resulting pre-Hilbert space. In the case at hand, with Haar measure on T^2, the inner product for L^2 sections can be expressed by

$$\langle f, g\rangle = \int_P f(\mathbf{x})\,\overline{g(\mathbf{x})}\,d\mathbf{x}$$

where P is a fundamental domain in $\mathbb{R}^2$, $d\mathbf{x}$ denotes Lebesgue measure on $\mathbb{R}^2$ and f and g are Borel functions satisfying the above relation specifying them as sections (with $\int_P |f|^2 d\mathbf{x} < \infty$, $\int_P |g|^2 d\mathbf{x} < \infty$) for some choice of F. The resulting Hilbert space is independent (up to equivalence) of the choices of F and of unitary structure and we shall denote it by $\mathcal{H}_\sigma$.

We shall denote the elements of T^2 locally by $\mathbf{x} = (x^0, x^1)$ with the implied physical interpretation given to the local parameters x^0 and x^1. To determine appropriate commutation relations we need to examine the effect of translations by elements (x^0, x^1) upon states (i.e., sections of the line bundle, denoted L, determined by σ). Now the translations by some fixed $\mathbf{x}_0 = (x_0^0, x_0^1)$ of a section of L yields a section of $\tau_{\mathbf{x}_0}^* L$ where

$$\tau_{\mathbf{x}_0}\colon T^2 \to T^2$$

denotes the action of translation by $\mathbf{x}_0$ and the asterisk superscript denotes bundle pull-back. So in order to obtain a full unitary automorphism of the space of sections of L induced by this translation we need to follow the act of translation by some unitary bundle isomorphism

$$\xi\colon \tau_{\mathbf{x}_0}^* L \xrightarrow{\sim} L$$

for each $\mathbf{x}_0$. Then the resulting operator would be given, for a section f of L, by a linear operator of the form

$$V(\xi, \mathbf{x}_0)(f) = \xi(\tau_{\mathbf{x}_0}^*(f))$$

Now it is not difficult to show that up to an arbitrary unitary phase factor the bundle isomorphism ξ is implemented by fiberwise multiplication by the function

$$\xi(\mathbf{x}) = \exp \pi i \sigma(\mathbf{x}, \mathbf{x}_0)$$

With this choice, and dropping the ξ reference on the left, we obtain finally

$V(\mathbf{x}_0)$ defined by

$$\begin{aligned} V(\mathbf{x}_0)(f)(\mathbf{x}) &= \xi(\mathbf{x})\tau^*_{\mathbf{x}_0}(f)(\mathbf{x}) \\ &= \exp \pi i \sigma(\mathbf{x},\mathbf{x}_0) f(\mathbf{x}+\mathbf{x}_0) \end{aligned}$$

Clearly, the operators $V(\mathbf{x})$ are unitary when L is given any of its pre-Hilbert structures and so they extend to unitary operators on $\mathcal{H}_\sigma$. These operators constitute a projective unitary representation of the group $\mathbb{R}^2$ belonging to a class first investigated by Cartier (Cartier, 1966). In particular, his results (Cartier, 1966, Section 12) apply to show that this representation is *irreducible* for any nonzero σ (and consequently *unique* up to unitary equivalence for each such σ by the Stone–von Neumann Theorem) since the standard lattice is principal relative to the nondegenerate form σ. We verify immediately that the $V(\mathbf{x})$ satisfy the Weyl commutation relations

$$V(\mathbf{x})V(\mathbf{y}) = \exp \pi i \sigma(\mathbf{x},\mathbf{y}) V(\mathbf{x}+\mathbf{y})$$

leading to

$$V(\mathbf{x})V(\mathbf{y}) = \exp 2\pi i \sigma(\mathbf{x},\mathbf{y}) V(\mathbf{y})V(\mathbf{x})$$

and, in infinitesimal form with

$$V(t\mathbf{x}) = e^{-itR(\mathbf{x})}, \qquad t \in \mathbb{R} \tag{24}$$

to

$$[R(\mathbf{x}), R(\mathbf{y})] = -2\pi i \sigma(\mathbf{x},\mathbf{y}) \tag{25}$$

with the usual domain conventions applying to the commutator.

Reverting to the interpretation in terms of connections, etc., one may abuse the notation and write, for some Hermitian connection D on L,

$$D^1 D = ig\, dx^0 \wedge dx^1$$

where g is real. Since $[D^1D/2\pi i]$ is an integral class in $\check{H}^2(T^2,\mathbb{C})$ we may, without loss of generality, put

$$\frac{g}{2\pi} = n g_0$$

for some $n \in \mathbb{Z}$ and some nonzero $g_0 \in \mathbb{R}$ which is the image of the generator of $\check{H}^2(T^2,\mathbb{R})$ establishing some isomorphism

$$\check{H}^2(T^2,\mathbb{R})(=\Lambda^2\mathbb{R}^2)\cong\mathbb{R}$$

Thus

$$D^1D=2\pi i n g_0 \; dx^0\wedge dx^1$$

and the associated "Lagrangian density" is proportional to

$$4\pi^2 n^2 g_0^2 \; dx^0\wedge dx^1$$

The Lagrangian achieves an extreme nonzero value when $n^2=1$, in which case we write

$$4\pi^2 g_0^2=h$$

where h is a certain constant associated with the minimum energy density.

Now if we had at the outset chosen the lattice $k\mathbb{Z}^2$ (where k is a real constant) then the considerations leading to the representation V and the conclusions concerning its irreducibility would remain unaffected except that σ would be replaced by σ/k^2 which is integer valued on the lattice $k\mathbb{Z}^2$ and with respect to which this lattice is principal. [Equivalently one could define a new representation by $\mathbf{x}\to V(k^{-1}\mathbf{x})$]. Conducting this rescaling with

$$\begin{aligned} k^2 &= g_0^{-2} \\ &= 4\pi^2/h \\ &= 2\pi/\hbar \end{aligned}$$

the commutation relations equation (25) begin to assume a more familiar form.

This rescaling of the dimensions of the fundamental domain of the standard lattice, when considered as the result of applying the automorphism of the coefficient field $\mathbb{R}$ induced by multiplication by g_0^{-1}, is tantamount to redefining the isomorphism $\check{H}^2(T^2,\mathbb{R})\cong\mathbb{R}$ used above to be the one in which the generator of the cohomology space is mapped to 1. With this generator represented by the differential form $dx^0\wedge dx^1$ the above isomorphism $\Lambda^2\mathbb{R}^2\to\mathbb{R}$ defines an alternating bilinear map σ on $\mathbb{R}^2$ for which

$$\sigma((1,0),(0,1))=1$$

With this choice for σ, equation (25) now yields

$$[R(1,0), R(0,1)] = -\hbar i \tag{26}$$

But from equation (24) the operator $R(1,0)$ is the infinitesimal generator of the one-parameter unitary group associated with time (i.e., x^0) translations. Consequently this operator should represent the observable associated with energy-momentum and we may write

$$R(1,0) \equiv P$$

Similarly $R(0,1)$ represents the observable associated with space (i.e., x^1) translations and we may write

$$R(0,1) \equiv Q$$

Equation (26) now assumes the even more familiar form

$$[P,Q] = -\hbar i$$

Since the representation is irreducible an application of the von Neumann commutator theorem shows that any observable belongs to the W^* algebra generated by P and Q and the usual paraphernalia of elementary quantum mechanics can be formally built up, though the states in this case are not represented by "wave functions" in the Schrödinger sense. In fact there is a relativistic interpretation which will be taken up elsewhere.

ACKNOWLEDGMENTS

Work for this paper was begun while visiting the Mathematical Institute, Oxford, during 1976–1977. The application to gauge invariance were inspired by lectures of Professor M. F. Atiyah on the geometry of Yang–Mills fields. For this, and for his generous hospitality, it is a pleasure to record my thanks.

REFERENCES

Atiyah, M. F., and Jones, J. D. S. (1978). *Communications in Mathematical Physics*, **61**, 97.

Bass, H. (1968). *Algebraic K-theory*, W. A. Benjamin, New York.

Bjorken, J. D., and Drell, S. D. (1964). *Relativistic Quantum Mechanics*, McGraw-Hill, New York.

Bjorken, J. D., and Drell, S. D. (1965). *Relativistic Quantum Fields*, McGraw-Hill, New York.

Bogoliubov, N. N. (1967). *Lectures on Quantum Statistics*, Vol. 1, MacDonald, London.

Cartier, P. (1966). "Quantum mechanical commutation relations and theta functions," Proceedings of Symposia in Pure Mathematics IX, American Mathematical Society, Providence, Rhode Island.

Kostant, B. (1970). "Quantization and unitary representation," in *Lectures in Modern Analysis and Applications III*, edited by C. T. Taam, Lecture Notes in Mathematics 170, Springer, Berlin.

Kuiper, N. (1953). "Sur les surfaces localement affines," Colloque de Géométrie Differentielle, Strasbourg.

Lang, S, (1965). *Algebra*, Addison-Wesley, Reading, Massachusetts.

MacLane, S. (1967). *Homology*, Springer-Verlag, New York.

Misner, S. W., Thorne, K. S., and Wheeler, J. A. (1973) *Gravitation*, W. H. Freeman, San Francisco.

Mulvey, C. J. (1976). *Mathematische zeitschrift*, **151**, 57.

Roman, P. (1965). *Advanced Quantum Theory*, Addison-Wesley, Reading, Massachusetts.

Scadron, M. D. (1979). *Advanced Quantum Theory*, Springer-Verlag, New York.

Segal, I. E. (1963). *Mathematical Problems of Relativistic Physics*, American Mathematical Society, Providence, Rhode Island.

Selesnick, S. A. (1979). "Rank one projective modules over certain Fourier algebras," *Applications of Sheaves*, edited by M. P. Fourman, C. J. Mulvey, and D. S. Scott, Lecture Notes in Mathematics 753, Springer, Berlin.

Śniatycki, J. (1974). *Journal of Mathematical Physics*, **15**, 619.

Swan, R. G. (1968). *Algebraic K-theory*, Lecture Notes in Mathematics 76, Springer-Verlag, New York.

Taylor, J. C. (1979). *Gauge Theories of Weak Interactions*, Cambridge University Press, Cambridge.

Vaisman, I. (1973) *Cohomology and Differential Forms*, Marcel Dekker, New York.

Yang, C. N., and Mills, R. L. (1954). *Physical Review*, **96**, 191.